

# A NOTE ABOUT INVARIANT TRANSFORMATIONS OF POLYNOMIAL INTEGER SEQUENCES.

LEONID BEDRATYUK

ABSTRACT. By using classical invariant theory approach an algorithm of finding invariant transformations of polynomial integer sequences is offered.

## 1. INTRODUCTION

Let  $\mathcal{A} = \{a_0, a_1, \dots\} = \{a_n\}$  be an integer sequence. A sequence  $\mathbf{F}(\mathcal{A}) = \{b_n\}$  where  $b_n$  is a polynomial of finite number of variables of  $\mathbb{Z}[\mathcal{A}] := \mathbb{Z}[a_0, a_1, \dots]$  is called a polynomial transformation of sequence  $\mathcal{A}$ . In the sequel, only polynomial transformations are considered. The composition  $\mathbf{F} \circ \mathbf{G} := \mathbf{F}(\mathbf{G}(\mathcal{A}))$  of two transformation  $\mathbf{F}$  and  $\mathbf{G}$  can be defined in a natural way. The transformation  $\mathbf{G}$  is called the inverse transformation of  $\mathbf{F}$ , and it is denoted by  $\mathbf{F}^{-1}$ , if for every sequence  $\mathcal{A}$  we have  $\mathbf{F}(\mathbf{G}(\mathcal{A})) = \mathcal{A}$ . A transformation  $\mathbf{F}$  is called  *$\mathbf{G}$ -invariant* if for every sequence  $\mathcal{A}$  we have  $\mathbf{F}(\mathbf{G}(\mathcal{A})) = \mathbf{F}(\mathcal{A})$ .

For instance, it is well known, see Layman [1], Spivey and Steil [2], that the Hankel transformation  $\mathbf{H}$  is  $\mathbf{B}_\mu$ -invariant. Here

$$\mathbf{B}_\mu(\mathcal{A}) = \left\{ b_n = \sum_{i=0}^n \binom{n}{i} a_i \mu^{n-i} \mid \mu \in \mathbb{Q} \right\},$$

denotes the  $\mu$ -binomial transformation and  $\mathbf{H}(\mathcal{A}) = \{h_n\}$ , where  $h_n$  is the determinant of Hankel matrix for the elements  $a_0, a_1, \dots, a_{2n}$ :

$$h_n = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-1} \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{vmatrix}.$$

This determinant is well known in the classical invariant theory as the *catalecticant* of binary form, see Olver [4]. Note that the transformation  $\mathbf{B}_\mu$  one may find in Hilbert's book [5, p. 25]. We can also prove that the Hankel transformation is  $\mathbf{B}_\mu$ -invariant by using classical invariant theory approach. In fact, let  $\mathcal{D}$  be the following differential operator acting on  $\mathbb{Z}[\mathcal{A}]$ :

$$\mathcal{D} = a_0 \partial_1 + 2a_1 \partial_2 + \cdots + na_{n-1} \partial_n + \cdots, \partial_i := \frac{\partial}{\partial a_i}.$$

Put

$$h'_n = \begin{vmatrix} b_0 & b_1 & b_2 & \cdots & b_n \\ b_1 & b_2 & b_3 & \cdots & b_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ b_{n-1} & b_n & b_{n+1} & \cdots & b_{2n-1} \\ b_n & b_{n+1} & b_{n+2} & \cdots & b_{2n} \end{vmatrix}.$$

Then Lie [6] implies

$$h'_n = h_n + \mathcal{D}(h_n)\mu + \mathcal{D}^2(h_n)\frac{\mu^2}{2!} + \cdots + \mathcal{D}^i(h_n)\frac{\mu^i}{i!} + \cdots$$

By applying the determinant derivative rule we obtain after some calculation that  $\mathcal{D}(h_n) = 0$  for all  $n$ . Therefore  $h'_n = h_n$ . This condition is exactly equivalent to the  $\mathbf{B}_\mu$ -invariance of Hankel transformation.

This motivates us to consider the following two general problems:

**Problem 1.** For given a transformation  $\mathbf{F}$  find all  $\mathbf{F}$ -invariant transformations.

**Problem 2.** For given a transformation  $\mathbf{F}$  find all transformations  $\mathbf{G}$  such that  $\mathbf{F}$  is  $\mathbf{G}$ -invariant transformation.

The aim of this paper is to develop an effective method for the solution of the above two problem for some special kinds of transformation. The origin of method came from the classical invariant theory and the theory of locally nilpotent derivations. We introduce the notion of exponential transformation and then prove that for such transformations the Problem 1 can always be solved. For instance, a transformation of the form  $\{b_n = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_{n-1} a_{n-1} + a_n \mid \alpha_i \in \mathbb{Z}\}$  is an exponential one.

In section 2 we give a short introduction to the theory of locally nilpotent derivations and offer algorithms to solve the Problems 1 and 2.

In section 3 we give another proof of  $\mathbf{B}_\mu$ -invariance the Hankel transformation and introduce several new  $\mathbf{B}_\mu$ -invariant transformations. Note, all of them came from classical invariant theory. Also we describe all  $\mathbf{B}_\mu$ -invariant transformations in term of derivation.

In section 4 we illustrate the theory by some examples.

## 2. DERIVATIONS AND AUTOMORPHISMS

Let  $\varphi : \mathbb{Z}[\mathcal{A}] \rightarrow \mathbb{Z}[\mathcal{A}]$  be a polynomial map. It means that  $\varphi$  is uniquely determined by the set of polynomials  $\{\varphi(a_n)\}$ . To any polynomial map  $\varphi$  we assign the transformation  $\{\varphi(a_n)\}$ . A polynomial map is said to be a *polynomial automorphism* if there is a polynomial map  $\psi$  such that  $\varphi(\psi(a_n)) = a_n$  for all  $n$ . Denote by  $\mathbb{Z}[\mathcal{A}]^\varphi$  the algebra of  $\varphi$ -invariants:

$$\mathbb{Z}[\mathcal{A}]^\varphi := \{f \in \mathbb{Z}[\mathcal{A}] \mid f(\varphi(a_0), \varphi(a_1), \dots, \varphi(a_n), \dots) = f(a_0, a_1, \dots, a_n, \dots)\}.$$

The following theorem be our main computing tool:

**Theorem 1.** *Let  $\varphi$  be a polynomial map and let  $\mathbf{F}(\mathcal{A}) = \{\varphi(a_n)\}$  be the corresponding integer transformation. Then the transformation*

$$\mathbf{G}(\mathcal{A}) = \{g_n \mid g_n \in \mathbb{Z}[\mathcal{A}]^\varphi\},$$

*is  $\mathbf{F}$ -invariant.*

The proof follows immediately from the above definitions.

In general, the problem of finding the algebra of  $\varphi$ -invariants is a difficult problem. But in the case when  $\varphi$  is so-called exponential automorphism this problem can be reduced to calculation of kernel of a derivation.

A *derivation* of the algebra  $\mathbb{Z}[\mathcal{A}]$  is a linear map  $D$  satisfying the Leibniz rule:

$$D(f_1 f_2) = D(f_1)f_2 + f_1 D(f_2), \text{ for all } f_1, f_2 \in \mathbb{Z}[\mathcal{A}].$$

A derivation  $D$  is called *locally nilpotent* if for every  $f \in \mathbb{Z}[\mathcal{A}]$  there is an  $n \in \mathbb{N}$  such that  $D^n(f) = 0$ . Let us recall that we consider only polynomials of finite number of variables. The subalgebra

$$\ker D := \{f \in \mathbb{Z}[\mathcal{A}] \mid D(f) = 0\},$$

is called the *kernel* of the derivation  $D$ .

Any derivation  $D$  is completely determined by the elements  $D(a_i)$ . A derivation  $D$  is called *linear* if  $D(a_i)$  is a linear form. A linear locally nilpotent derivation is called a *Weitzenböck derivation*. The Weitzenböck derivation defined by  $\mathcal{D}(a_i) = ia_{i-1}$  is called the *basic Weitzenböck derivation*. The kernel of basic Weitzenböck derivation is well known. There exists an isomorphism between the algebra  $\ker \mathcal{D}$  and the algebra of covariant of binary form, a major object of research in the classical invariant theory of the 19th century. Here are a few examples of covariants: the discriminant, the resultant, the jacobian, the hessian, the catalecticant and the transvectant. The following theorem gives a description of the algebra  $\ker \mathcal{D}$  for a fixed number of involved variables:

**Theorem 2.** *The kernel of the basic Weitzenböck derivation  $\mathcal{D}$  of  $\mathbb{Q}[a_0, a_1, \dots, a_n]$  is finitely generated algebra and*

$$\ker \mathcal{D} = \mathbb{Q}[z_2, z_3, \dots, z_n][a_0, a_0^{-1}] \cap \mathbb{Q}[a_0, a_1, \dots, a_n],$$

where

$$z_m = \sum_{i=0}^{m-2} (-1)^i \binom{m}{i} a_{m-i} a_1^i a_0^{m-i-1} + (m-1)(-1)^{m+1} a_1^m.$$

It is a classical result due to Cayley, see [3], page 164. The modern proof one may find in [9], [7].

How to find the kernel of arbitrary linear locally nilpotent derivation  $D$ ? Let us consider the vector space (over  $\mathbb{Q}$ ) with the basis  $\langle a_0, a_1, \dots, a_n \rangle$ . Suppose that there exists an isomorphism  $\Psi : \langle a_0, a_1, \dots, a_n \rangle \rightarrow \langle a_0, a_1, \dots, a_n \rangle$  such that  $\Psi \mathcal{D} = D \Psi$ . It is obviously that  $\ker D = \Psi(\ker \mathcal{D})$ , i.e.

$$\ker D = \mathbb{Z}[\Psi(z_2), \Psi(z_3), \dots, \Psi(z_n)][\Psi(a_0), \Psi(a_0)^{-1}] \cap \mathbb{Z}[a_0, a_1, \dots, a_n].$$

Such an isomorphism  $\Psi$  is called the  $(\mathcal{D}, D)$ -*intertwining* isomorphism. Therefore, to describe the kernel of arbitrary Weitzenböck derivation  $D$  it is enough to know the explicit form of any  $(\mathcal{D}, D)$ -*intertwining* isomorphism.

An automorphism  $\varphi$  is called *exponential* if there exists a locally nilpotent derivation  $D$  such that

$$\varphi = \exp(D) = D^0 + D + \frac{1}{2!} D^2 + \dots.$$

For instance, any automorphism of the form

$$\varphi(a_n) = a_n + f(a_0, a_1, \dots, a_{n-1}), f \in \mathbb{Z}[a_0, a_1, \dots, a_{n-1}],$$

is exponential, see Drensky and Yu [8]. For any exponential automorphism  $\varphi = \exp(D)$  the following statement holds:

$$\mathbb{Q}[a_0, a_1, \dots, a_n]^\varphi = \ker D.$$

For integer transformations we may introduce an analogue of above notations.

**Definition 2.1.** *The transformation  $D(\mathbf{F}(\mathcal{A})) := \{D(f_n)\}$  is called the  $D$ -derivative of transformation  $\mathbf{F} = \{f_n \mid f_n \in \mathbb{Z}[\mathcal{A}]\}$ .*

**Definition 2.2.** *A transformation  $\mathbf{F}$  is called exponential if there exists a locally nilpotent derivation  $D$  such that*

$$\mathbf{F}(\mathcal{A}) = \exp D(\mathcal{A}).$$

We may rewrite the Theorem 1 in the following way

**Theorem 3.** Suppose that a transformation  $\mathbf{F}$  be exponential and  $\mathbf{F}(\mathcal{A}) = \exp D(\mathcal{A})$ . Then a transformation  $\mathbf{G}$  is  $\mathbf{F}$ -invariant if  $D(\mathbf{G}(\mathcal{A})) = \mathbf{0}$ ,  $\mathbf{0} = \{0, 0, 0, \dots\}$ .

The Weitzenböck derivations are related with some special transformations by the following theorem:

**Theorem 4.** The transformation  $F(\mathcal{A}) := \left\{ a_n + \sum_{i=0}^{n-1} \alpha_i a_i \mid \alpha_i \in \mathbb{Z} \right\}$  is exponential and  $F(\mathcal{A}) = \exp D(\mathcal{A})$ , where the derivation  $D$  is a Weitzenböck derivation defined by

$$D(f) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} E^i(f),$$

and  $E = \varphi - \mathbf{1}$  is a locally nilpotent map.

The proof follows from [7], Proposition 2.1.3.

Thus this yields an algorithm for solve the Problem 1 in a case when the transformation  $\mathbf{F}(\mathcal{A}) = \{b_n\}$  has the special form

$$b_n = a_n + \sum_{i=0}^{n-1} \alpha_i a_i, \alpha_i \in \mathbb{Z}.$$

In this case for the corresponding automorphism  $\varphi(a_n) = a_n + \sum_{i=0}^{n-1} \alpha_i a_i$  we find the explicit form of the Weitzenböck derivation  $D$  such that  $\varphi = \exp(D)$  (Theorem 3). After that we find the  $(\mathcal{D}, D)$ -intertwining automorphism  $\Psi$  and obtain the kernel  $\ker D = \Psi(\ker \mathcal{D})$ . Then an arbitrary sequence of kernel elements defines  $\mathbf{F}$ -invariant transformation (Theorem 1.).

To solve the Problem 2 we find a locally nilpotent derivation  $D$  such that  $b_n \in \ker D$ . It can be done by the method of indefinite coefficients. So we define the automorphism  $\varphi = \exp D$ . and the transformation  $\mathbf{G}(\mathcal{A}) = \{b_n = \varphi(a_n)\}$ . Thus the transformation  $\mathbf{F}$  is  $\mathbf{G}$ -invariant by Theorem 1. Note that the method does not work in the case when  $b_n = f(b_0, b_1, \dots, b_{n-1})$ , particularly if  $\varphi$  is a linear automorphism. Indeed, the conditions  $D(b_0) = D(b_1) = 0, \dots, D(b_{n-1}) = 0$  follows that  $D(b_n) = 0$ . Therefore the derivation  $D$  is trivial.

### 3. THE $\mu$ -BINOMIAL TRANSFORMATIONS.

We use the developed techniques to get another proof of the well known result, see [1],[2].

**Theorem 5.** The Hankel transformation  $\mathbf{H}$  is  $\mathbf{B}_\mu$ -invariant.

*Proof.* We follows the above algorithm. The corresponding to  $\mathbf{B}_\mu$  automorphism  $\varphi_\mu$  has the form:

$$\varphi_\mu(a_n) = \sum_{i=0}^n \binom{n}{i} a_i \mu^{n-i}.$$

For the basic Weitzenböck derivation  $\mathcal{D}$  we have

$$\begin{aligned} \exp(\mu \mathcal{D})(a_n) &= \sum_{i \geq 0} \frac{1}{i!} (\mu \mathcal{D})^i (a_n) = \sum_{i=0}^n \frac{n(n-1) \dots (n-(i-1))}{i!} \mu^i a_{n-i} = \\ &= \sum_{i=0}^n \binom{n}{i} \mu^i a_{n-i} = \sum_{i=0}^n \binom{n}{i} a_i \mu^{n-i}. \end{aligned}$$

Thus  $\varphi_\mu = \exp(\mu\mathcal{D})$ . It follows that the transformation  $\mathbf{B}_\mu$  is exponential, i.e.  $\mathbf{B}_\mu = \exp(\mu\mathcal{D})\mathcal{A}$ . Since the catalecticant belongs to the kernel of derivation  $\mathcal{D}$  we have that  $\mathcal{D}(\mathbf{H}) = \mathbf{0}$ . Then by Theorem 3 we obtain that the transformation  $\mathbf{H}$  is  $\mathbf{B}_\mu$ -invariant.

The map  $\exp(\mu\mathcal{D}) : Q[a_0, a_1, \dots, a_n] \rightarrow Q[\mu, a_0, a_1, \dots, a_n]$  is a ring homomorphism, see [7], Proposition 1.2.24. It follows that  $\varphi_{\mu_1+\mu_2} = \varphi_{\mu_1} \circ \varphi_{\mu_2}$ . Therefore  $\varphi_\mu \circ \varphi_{-\mu}$  is the identity map and  $\mathbf{B}_\mu^{-1} = \mathbf{B}_{-\mu}$ . It follows immediately that the inverse transformation  $\mathbf{B}_\mu^{-1}$  is also  $\mathbf{H}$ -invariant transformation.  $\square$

As a corollary of identity  $\varphi_{\mu_1+\mu_2} = \varphi_{\mu_1} \circ \varphi_{\mu_2}$ , we obtain that  $\mathbf{B}_\mu = \mathbf{B}_1^\mu$  and  $\mathbf{B}_\mu^{-1} = \mathbf{B}_{-1}^\mu$ .

All  $\mathbf{B}_\mu$ -invariant transformation forms a group, see French [10]. The identity  $\varphi_{\mu_1+\mu_2} = \varphi_{\mu_1} \circ \varphi_{\mu_2}$  implies that the group  $(\mathbb{Z}, +)$  is a subgroup of those group.

The following theorem is a solution of the Problem 1 for the  $\mu$ -binomial transformation:

**Theorem 6.** *A transformation  $\mathbf{F}$  is  $\mathbf{B}_\mu$ -invariant if and only if  $\mathcal{D}(\mathbf{F}(\mathcal{A})) = \mathbf{0}$ .*

Below we offer some of Hanker-type transformations which arise from the classical invariant theory. Note that all of that transformation are  $\mathbf{B}_\mu$ -invariant and  $\mathbf{B}_\mu^{-1}$ -invariant.

**3.1. Cayley transformation.** Put  $\mathbf{CAYLEY}(\mathcal{A}) = \{b_n\}$ ,  $b_0 = a_0$ ,

$$b_n = \sum_{i=0}^{n-2} (-1)^i \binom{n}{i} a_{n-i} a_1^i a_0^{n-i-1} + (n-1)(-1)^{n+1} a_1^n.$$

**3.2. Transvectant transformation.** Let  $\mathcal{A} = \{a_n\}, \mathcal{C} = \{c_n\}$  be two sequences. The transformation  $\mathbf{TR}(\mathcal{A}, \mathcal{C}) = \{b_n\}$ ,

$$b_n = \sum_{i=0}^n (-1)^i \binom{n}{i} a_i c_{n-i},$$

is called the transvectant transformation. We have

$$\mathbf{Tr}(\mathbf{B}_\mu(\mathcal{A}), \mathbf{B}_\mu(\mathcal{C})) = \mathbf{Tr}(\mathcal{A}, \mathcal{C}).$$

In the case  $\mathcal{C} = \mathcal{A}$  we get

$$b_n = \sum_{i=0}^n (-1)^i \binom{n}{i} a_i a_{n-i}.$$

**3.3. Resultant transformation.** Let  $\mathcal{A} = \{a_n\}, \mathcal{C} = \{c_n\}$  be two sequences. The transformation  $\mathbf{RES}(\mathcal{A}, \mathcal{C}) = \{b_n\}$  where  $b_n$  is the leading coefficient of resultant of the polynomials

$$P_n(\mathcal{A}) = \sum_{i=0}^n \binom{n}{i} a_i X^{n-i}, P_n(\mathcal{C}) = \sum_{i=0}^n \binom{n}{i} c_i X^{n-i},$$

is called the *resultant transformation*.

**3.4. Discriminant transformation.** The transformation  $\mathbf{DISCR}(\mathcal{A}) = \{b_n\}$  where  $b_n$  is the discriminant of the polynomial

$$P_{n+2}(\mathcal{A}) = \frac{1}{(n+2)^{n+2}} \sum_{i=0}^{n+2} a_i \binom{n+2}{i} X^{n+2-i},$$

is called the *discriminant transformation*.

## 4. EXAMPLES.

**4.1. Transformation**  $\text{PSUM}(\mathcal{A}) = \{b_n = a_0 + a_1 + \dots + a_n\}$ . The corresponding locally nilpotent derivation (see Theorem 3) has the form

$$D(a_n) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} E^i(a_n).$$

We have

$$\begin{aligned} E(a_0) &= 0, E(a_n) = b_n - a_n = a_0 + a_1 + a_2 + \dots + a_{n-1}, \\ E^2(a_n) &= \sum_{i=0}^{n-1} E(a_i) = \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} a_j = \sum_{i=0}^{n-2} (n-1-i) a_i. \end{aligned}$$

By induction we obtain  $E^i(a_n) = \sum_{k=0}^{n-i} \binom{n-i-1}{i-1} a_k$ . Then

$$D(a_n) = \sum_{i=1}^n \frac{(-1)^{i+1}}{i} \sum_{k=0}^{n-i} \binom{n-i-1}{i-1} a_k = \sum_{k=0}^{n-1} \left( \sum_{i=0}^{n-1-k} \frac{(-1)^i}{i+1} \binom{n-1-k}{i} \right) a_k = \sum_{k=0}^{n-1} \frac{a_k}{n-k}.$$

Let us find  $(\mathcal{D}, D)$ -intertwining transformation  $\Psi$ . We show that

$$\Psi(\mathcal{A}) = \left\{ \Psi(a_n) = \sum_{k=0}^n (-1)^{n+k} k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} a_k, \Psi(a_0) = a_0 \right\},$$

where  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  is the Stirling numbers of second kind. In fact,

$$\begin{aligned} D(\Psi(a_n)) &= D \left( \sum_{k=0}^n (-1)^{n+k} k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} a_k \right) = \sum_{k=0}^n (-1)^{n+k} k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \sum_{i=0}^{k-1} \frac{a_i}{k-i} a_i = \\ &= \sum_{i=0}^{n-1} \sum_{j=i+1}^n (-1)^{n+j} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} \frac{j!}{j-i} a_i = n \sum_{i=0}^{n-1} (-1)^{n-1+i} \left\{ \begin{matrix} n-1 \\ i \end{matrix} \right\} i! a_i = \Psi(\mathcal{D}(a_n)). \end{aligned}$$

Therefore now we may construct PSUM-invariant transformation by using already known  $\mathbf{B}_\mu$ -invariant transformations. For instance, the transformation

$$\begin{aligned} \Psi(\mathbf{H}(\mathcal{A})) &= \{a_0, -a_1^2 - a_1 a_0 + 2 a_2 a_0, -4 a_1 a_2 a_0 + 24 a_1 a_2 a_3 + 24 a_0 a_1 a_3 + 48 a_0 a_2 a_4 - 8 a_2^3 - \\ &\quad - 8 a_0 a_2^2 - 12 a_1 a_2^2 - 36 a_0 a_3^2 - 4 a_1^2 a_2 - 24 a_1^2 a_4 + 24 a_1^2 a_3 - 24 a_0 a_1 a_4, \dots\}, \end{aligned}$$

is PSUM-invariant.

**Problem.** What is the explicit form of  $\Psi(\mathbf{F})$  for  $\mathbf{F} \in \{\mathbf{CAYLEY}, \mathbf{H}, \mathbf{RES}, \mathbf{DISCRIM}, \mathbf{TR}\}$ ?

**4.2. The Transformation**  $\{b_n = a_n + a_{n-1}\}$ . We have  $E(a_n) = \varphi(a_n) - a_n = a_{n-1}$  and

$$D(a_n) = \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} E^i(a_n) = \sum_{i=1}^n \frac{(-1)^{i+1}}{i} a_{n-i}.$$

Let

$$\Psi(a_0) = a_0, \Psi(a_n) = c_{n,1} a_1 + c_{n,2} a_2 + \dots + c_{n,n} a_n.$$

The  $(\mathcal{D}, D)$ -intertwining map satisfies the conditions  $D(\Psi(a_n)) = \Psi(\mathcal{D}(a_n))$ . After a routine calculation we get

$$\Psi(a_n) = \sum_{i=1}^n i! \left\{ \begin{matrix} n \\ i \end{matrix} \right\} a_i.$$

**4.3. Transformation**  $\text{DIFF}(\mathcal{A}) = \{b_n = a_n - a_{n-1}\}$ . We have  $E(a_n) = -a_{n-1}$  and  $E^i(a_n) = (-1)^i a_{n-i}$ . Then

$$D(a_n) = \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} E^i(a_n) = \sum_{i=1}^n \frac{(-1)^{i+1}}{i} (-1)^i a_{n-i} = - \sum_{i=1}^n \frac{a_{n-i}}{i}.$$

The  $(\mathcal{D}, D)$ -intertwining map has the form

$$\Psi(a_n) = \sum_{i=1}^n (-1)^i i! \left\{ \begin{matrix} n \\ i \end{matrix} \right\} a_i.$$

**4.4. The transformation  $\mathbf{F}$**   $= \{b_n = \sum_{i=0}^{2n} (-1)^i a_i a_{2n-i}\}$ . Let us try to solve the Problem 2 for this transformation. To do it we have to find a suitable locally nilpotent derivation that satisfies the conditions  $D(b_n) = 0$ . Let us consider the following locally nilpotent derivation  $D(a_i) = a_{i-1}$ . In the author's paper [11] it is proved that for the derivation  $D$  holds  $D(b_n) = 0$ . Let us calculate the exponential automorphism  $\varphi = \exp D$ . We have

$$\varphi(a_n) = D^0(a_n) + D(a_n) + \frac{1}{2!} D^2(a_n) + \cdots = a_n + a_{n-1} + \frac{1}{2} a_{n-2} + \frac{1}{n!} a_0.$$

Define a (rational) transformation  $\mathbf{G}$  by  $\mathbf{G}(\mathcal{A}) := \{\varphi(a_n)\}$ . Then  $\mathbf{F}(\mathbf{G}(\mathcal{A})) = \mathbf{F}(\mathcal{A})$ .

## REFERENCES

- [1] Layman, John W. The Hankel transform and some of its properties, J. Integer Seq. 4 (2001), No.1, Art. 01.1.5, 11 p.
- [2] Michael Spivey, Laura L. Steil, The  $k$ -Binomial Transforms and the Hankel Transform, J. Integer Seq. 9(2006), No.1, Art. 06.1.1, 19 p.
- [3] Glenn O. Treatise on theory of invariants/O. Glenn.–Boston,1915.–312P.
- [4] Olver P. Classical invariant theory/P. Olver.–Cambridge: Cambridge University Press,1999.–280 p.
- [5]
- [6] Lie S. Theorie der Transformationsgruppen. Erster Abschnitt. Unter Mitwirkung von F. Engel bearbeitet,–1888.–Leipzig:Teubner.–658S.  
Hilbert D. Theory of algebraic invariants/ D. Hilbert.–Cambridge University Press,1993.–191 p.
- [7] van den Essen A. Polynomial automorphisms and the Jacobian conjecture/A. van den Essen.–Basel: Birkhäuser. –2000.–329 p.
- [8] Drensky V., Yu J.-T., Exponential automorphism of polynomial algebras, Comm. Algebra,26 (1998), 2977–2985.
- [9] Nowicki A. Polynomial derivation and their Ring of Constants.–UMK: Torun,–1994.
- [10] French, Christopher Transformations preserving the Hankel transform, J. Integer Seq. 10, No. 7, Article 07.7.3, 14 p., (2007).
- [11] Bedratyuk, L.P. Kernels of derivations of polynomial rings and Casimir elements, Ukr. Mat. Zh. 62, No. 4, 435-452 (2010);